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THE PRESSURE EXERTED BY A STAMP OF CIRCULAR CROSS-SECTION ON AN ELASTIC HALF-SPACE*

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A solution of the problem of a circular stamp in its exact formulation, i.e. without simplifying assumptions regarding satisfaction of the boundary conditions and Laplace's equation, is obtained. A method of solving three-dimensional contact problems of the theory of elasticity due to Mossakovskii is used, and the solution obtained is compared with the solution given in /1/.

As we know /2/, the problem of the pressure exerted by a stamp of circular cross-section reduces, in the case of axial symmetry, to determining the normal derivative $F_z'(\rho, 0)$ in the region of contact, and the function $F(\rho, z)$ harmonic in the half-space and vanishing at infinity, which satisfies the following boundary conditions:

$$F_z'(\rho, 0) = 0,\ 0 < \rho < a,\ b < \rho < \infty,\ F(\rho, 0) = f(\rho)\quad a < \rho < \infty \tag{1}$$

where a and b denote the inner and outer radius of the annulus, ρ is the polar radius, and $z = f(\rho)$ is the equation of the stamp surface (the z axis is directed into the elastic half-space).

The pressure under the stamp $P(\rho)$ is given by the formula

$$P(\rho) = 1/2E(1 - \nu^2)^{-1} F_z'(\rho, 0),\ a < \rho < b$$

where E is the modulus of elasticity and ν is Poisson's ratio.

In the general case we must assume that

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$$f(\rho) = f_1(\rho) + f_2(\rho),\ f_1(\rho) = a_0 + a_1\rho + \ldots,\ f_2(\rho) = a_{-1}\rho^{-1} + a_{-2}\rho^{-2}$$

Such a representation of the function $f(\rho)$ is obviously unique, also $f_1(\rho)$ can be continued up to the zero and $f_2(\rho)$ up to infinity.

Let us introduce two functions $F_k(\rho, z)$ $(k = 1, 2)$ harmonic in the elastic half-space and such, that

$$F_1(\rho, z) + F_2(\rho, z) = F(\rho, z) \tag{2}$$
$$F_1(\rho, 0) = f_1(\rho),\ 0 < \rho < b;\ F_2(\rho, 0) = f_2(\rho),\ a < \rho < \infty$$

Then the boundary conditions (1) will take the form

$$F_{1z}'(\rho, 0) + F_{2z}'(\rho, 0) = 0,\ 0 < \rho < a,\ b < \rho < \infty \tag{3}$$
$$F_1(\rho, 0) = f_1(\rho),\ 0 < \rho < b;\ F_2(\rho, 0) = f_2(\rho),\ a < \rho < \infty$$

The functions $F_k(\rho, z)$ harmonic in the half-space $z \leqslant 0$ can be written in the form

$$F_k(\rho, z) = J\Phi_k(t)\, e^{tz} J_0(t\rho)\, dt \tag{4}$$

where $\Phi_k(t)$ are functions as yet unknown, and J_0 is a Bessel function of zero order. Here and henceforth the integration with respect to t will be carried out from 0 to ∞.

Let us put $z = 0$ in formulas (4) and replace the Bessel functions by contour integrals. Changing the order of integration and introducing the notation $\varphi_k(s) = J\Phi_k(t)\, t^s dt$ we obtain a system of integral equations for determining the functions $\varphi_k(s)$ (from now on the integration in s will be carried out from $c - \iota\infty$ to $c + \iota\infty$)

$$\frac{1}{2\pi\iota} J 2^{-s}\varphi_k(s)\frac{\Gamma(^1/_2 - {^1/_2}s)}{\Gamma(^1/_2 + {^1/_2}s)}\rho^{s-1}\, ds = \begin{cases} f(\rho), & k = 1,\ 0 < \rho < b \\ 0, & k = 2,\ a < \rho < \infty \end{cases} \tag{5}$$
$$\frac{1}{2\pi\iota} J 2^{1-s}(\varphi_1(s) + \varphi_2(s))\frac{\Gamma(1 - {^1/_2}s)}{\Gamma(^1/_2 s)}\rho^{s-2}\, ds = 0,\ 0 < \rho < a,\ b < \rho < \infty$$

Let two harmonic functions $y < 0$ symmetrical in x, be given in the half-plane $u_k(x, y)$

$$u_k(x, y) = J\psi_k(t) \cos xt e^{ty} dt \tag{6}$$

where $\psi_k(t)$ are unknown functions. Putting $y = 0$ in formula (6), substituting into it the value of $\cos xt$ in the form of a contour integral and changing the order of integration, we obtain

$$u_k(x, 0) = \frac{1}{2\pi\iota} J \sqrt{\pi}\, 2^{-s-1} G_k(s)\frac{\Gamma(-{^1/_2}s)}{\Gamma(^1/_2 + {^1/_2}s)} x^s\, ds \tag{7}$$
$$G_k(s) = J\psi_k(t)\, t_s dt$$

Let us require that

$$F_{1z}'(\rho, 0) = \rho^{-1}u_{1x}'(\rho, 0),\ F_{2z}'(\rho, 0) = \rho^{-1}u_{2y}'(\rho, 0) \tag{8}$$

Having established such a relation between the planar and spatial functions, we can find the pressure under the stamp directly from the plane problem without using integral relationships.

Using relations (8) we obtain a relation connecting the kernels $G_k(s)$ and $\varphi_k(s)$.

Substituting the result into Eqs.(5) and using the following transformation formulas:

$$\int_0^x \rho^{2\alpha-1}(x^2 - \rho^2)^{\beta-1}\, d\rho = {^1/_2}\beta(\alpha, \beta)\, x^{2\alpha+2\beta-2}$$
$$\int_x^\infty \rho^{2\alpha-2\beta+1}(\rho^2 - x^2)^{\beta-1}\, d\rho = {^1/_2}\beta(\alpha, \beta)\, x^{-2\alpha}$$

we obtain the following system of equations:

$$\frac{1}{2\pi\iota} J \sqrt{\pi}\, 2^{-s} G_1(s)\frac{\Gamma(^1/_2 - {^1/_2}s)}{\Gamma(^1/_2 s)}\rho^{s-1}\, ds = \frac{f(\rho)}{4\pi},\quad 0 < \rho < b$$
$$\frac{1}{2\pi\iota} J \sqrt{\pi}\, 2^{-s}\left[G_1(s)\frac{\Gamma(1 - {^1/_2}s)}{\Gamma(^1/_2 + {^1/_2}s)} - G_2(s)\frac{\Gamma(^1/_2 - {^1/_2}s)}{\Gamma(^1/_2 s)}\right]\rho^{s-1}\, ds = 0 \tag{9}$$
$$0 < \rho < a,\quad b < \rho < \infty,$$
$$\frac{1}{2\pi\iota} J \sqrt{\pi}\, 2^{-s} G_2(s)\frac{\Gamma(1 - {^1/_2}s)}{\Gamma(^1/_2 + {^1/_2}s)}\rho^{s-1}\, ds = 0,\quad a < \rho < \infty$$

Taking into account relations (7), we obtain the following boundary conditions for the functions $u_k(x, 0)$:

$$u_{1x}'(x,\ 0) + u_{2y}'(x,\ 0) = 0,\ \ 0 < x < a,\ \ b < x < \infty$$
$$u_{2x}'(x,\ 0) = 0,\ \ a < x < \infty$$
$$u_{1y}'(x, 0) = \frac{1}{4\pi}\frac{d}{dx}\int_0^x \frac{z\,dz}{\sqrt{x^2 - z^2}}\cdot\frac{d}{dz}\int_0^z \frac{\rho f(\rho)\,d\rho}{\sqrt{z^2-\rho^2}},\quad 0 < x < b \tag{10}$$

In the case of a stamp with a flat base, we can write

$$f_1(\rho) = c,\ f_2(\rho) = 0$$

Let us map the region $y < 0$ onto the inside of a circle lying in the plane $\omega = \xi + i\eta$, using a single-valued analytic function $\omega = R(z)$. The latter relation establishes a one-to-one correspondence between the points x of the contour $y = 0$, and points $t = e^{i\varphi}$ of the circle $t = R(x)$. At the same time we establish one-to-one correspondence between the points $a, -a, b, -b$ and the points of a circle of the following form:

$$a \Rightarrow e^{i\theta_1},\quad -a \Rightarrow e^{-i\theta_1},\quad b \Rightarrow e^{i\theta},\quad -b \Rightarrow e^{-i\theta_2}\quad (\theta_2 = \pi - \theta_1)$$

The unknown function mapping the lower half-plane onto the inside of the circle will have the form

$$\omega = i\sqrt{ab}\,(1 - z)/(1 + z)$$

The boundary of the half-plane $y = 0$ will be transformed under this mapping into the circle $|e^{i\varphi}| = 1$, and

$$\varphi = 2\ \mathrm{arctg}\,(x/\sqrt{ab})$$

Let us introduce certain analytic functions

$$\Phi_k(\omega) = \Sigma A_{kn}\omega^{n-1},\quad k = 1, 2$$

where A_{kn} are constants (from now on the summation will be carried out from $n = 1$ to ∞).

Let the functions u_{kx}, u_{ky} be connected with Φ_k in the following manner:

$$\Phi_k(t) = u_{ky}'(x,\ 0) + iu_{kx}'(x,\ 0)$$

Then we can represent the functions $u_{ky}'(x, 0), u_{kx}'(x, 0)$ by certain trigonometric series

$$u_{kx}'(x, 0) = \Sigma A_{kn}\sin(n-1)\varphi \tag{11}$$
$$u_{ky}'(x, 0) = \Sigma A_{kn}\cos(n-1)\varphi$$

The functions $u_k(x, 0)$ have singularities at the points $a, -a, b, -b$ (conditions (10)) respectively, and the series (11) will converge slowly.

The following example will simplify the calculations by increasing the rate of convergence of the series. To do this we introduce a new series connected with (11) by the relation

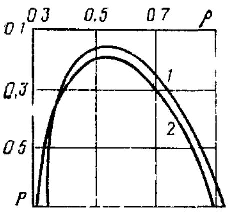
$$\Sigma C_{kn}t^{n-1} = \Sigma A_{kn}t^{n-1}(t - e^{i\theta_1})(t - e^{-i\theta_1})(t - e^{-i(\pi-\theta_1)})(t - e^{i(\pi-\theta_1)}) = \Sigma A_{kn}t^{n-1}\Delta$$
$$\Delta = 2(\cos 2\varphi - \cos 2\theta_1)$$

Taking into account the manipulations which have been carried out, we obtain

$$u_{kx}'(x, 0) = \Sigma C_{kn}/\Delta \sin(n-1)\varphi,\quad u_{ky}'(x, 0) = \Sigma C_{kn}/\Delta\cos(n-1)\varphi$$

The constants C_{kn} are found using formulas (10) with help of the method of least squares. Conditions (8) yield the pressure under the stamp

$$P(\rho) = {}^1/_2E\,(1 - \nu^2)^{-1}\rho^{-1}(u_{1x}'(\rho, 0) + u_{2y}'(\rho, 0)) \tag{12}$$



The pressure distribution under the stamp with a flat base which was solved for $a/b = 0.3$ $(b = 1)$ using formula (12), apart from the multiplier ${}^1/_2E\,(1 - \nu^2)^{-1}$, is shown in the figure. Curve *1* is constructed using the present method, and curve *2* using the method described in /1/.

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